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Differential invariants and group foliation for the complex Monge–Ampère equation

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Abstract

We apply the method of group foliation to the complex Monge–Ampère equation (CMA_2) with the goal of establishing a regular framework for finding its non-invariant solutions. We employ the infinite symmetry subgroup of the equation, the group of unimodular biholomorphisms, to produce a foliation of the solution space into leaves which are orbits of solutions with respect to the symmetry group. Accordingly, CMA_2 is split into an automorphic system and a resolvent system which we derive in this paper. This is an intricate system and here we make no attempt to solve it in order to obtain non-invariant solutions.

We obtain all differential invariants up to third order for the group of unimodular biholomorphisms and, in particular, all the basis differential invariants. We construct the operators of invariant differentiation from which all higher differential invariants can be obtained. Consequently, we are able to write down all independent partial differential equations with one real unknown and two complex independent variables which keep the same infinite symmetry subgroup as CMA_2. We prove explicitly that applying operators of invariant differentiation to third-order invariants we obtain all fourth-order invariants. At this level we have all the information which is necessary and sufficient for group foliation.

We propose a new approach in the method of group foliation which is based on the commutator algebra of operators of invariant differentiation. The resolving equations are obtained by applying this algebra to differential invariants with the status of independent variables. Furthermore, this algebra together with Jacobi identities provides the commutator representation of the resolvent system. This proves to be the simplest and most natural way of arriving at the resolving equations.

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1. Introduction

An important problem for partial differential equations admitting a Lie symmetry group is the construction of solutions that admit no continuous symmetries. According to the terminology of group analysis [1, 2] these are non-invariant solutions. In particular, the construction of the metric for $K3$ which is the most important gravitational instanton requires non-invariant solutions of the complex Monge–Ampère equation. On a complex differentiable manifold M of dimension two the complex Monge–Ampère equation, hereafter denoted by CMA_2, is given by

$$\det u_{i\bar{j}} = \kappa \tag{1}$$

$$u_{i\bar{j}} \stackrel{def}{=} \frac{\partial^2 u}{\partial \zeta^i \, \partial \bar{\zeta}^j} \tag{2}$$

where subscripts denote partial derivatives and a bar denotes complex conjugation. Gravitational instantons are given by Kähler metrics

$$\mathrm{d}s^2 = u_{i\bar{j}} \, \mathrm{d}\zeta^i \, \mathrm{d}\bar{\zeta}^j \tag{3}$$

with $\kappa = +1$, the elliptic case of equation (1), where summation over the two values of both unbarred and barred indices is understood. Kähler metrics with self-dual curvature are Ricci-flat solutions of the Einstein field equations with Euclidean signature [3].

In our opinion the method of group foliation provides a regular framework for finding non-invariant solutions of differential equations. We develop this method further by introducing and extensively using the commutator algebra of the operators of invariant differentiation. In this paper we apply this method to CMA_2 with the idea of setting up a framework for constructing its solutions that admit no continuous symmetries.

An alternative approach which is based on conservation laws and associated 'potential symmetries' was constructed by Bluman [4]. However, in a multi-dimensional problem such as CMA_2 there is no regular way of obtaining gauge constraints for potentials and hence for the existence of potential symmetries [5].

In section 2 we discuss the main features of the method of group foliation as applied to CMA_2 and the most essential infinite symmetry group of unimodular biholomorphic transformations that we shall use. The differential invariants up to and including third order are constructed explicitly in section 3. Operators of invariant differentiation are presented in section 4. Except for the basis invariants, higher-order differential invariants are generated by the application of these operators to lower-order invariants. At third order we are able to fix all of the basis differential invariants. We derive the commutator algebra of vector fields formed by the operators of invariant differentiation. This is used to obtain important relations between fourth-order invariants. In section 5 we split the set of differential invariants that survive on the solution manifold of CMA_2 into two sets where one set consists of differential invariants that will be considered as new independent variables and the remaining differential invariants are considered as functions of the first set. This fixes the general form of the automorphic system. After a suitable redefinition of the operators of invariant differentiation the commutator algebra simplifies. Applying this algebra to differential invariants treated as independent variables we obtain the major part of the resolvent system. The remaining resolving equations are obtained by the integrability conditions between the automorphic equations and CMA_2. We discover that the commutator algebra of the operators of invariant differentiation and its Jacobi identities are equivalent to the *complete set* of resolving equations for CMA_2. The resolving equations form an intricate system. Even though a search for its solutions is the main motivation for this work, we shall not consider it here.

2. Group foliation

The approach we shall follow is based on the theory of 'group splitting' which was originally an idea of Lie [1], developed by Vessiot [6], and resurrected in modern form by Ovsiannikov [2]. According to this method we use a symmetry subgroup of the equation, preferably an infinite one, for producing a foliation of the equation into an *automorphic* system and a *resolvent* system. After that the real problem is reduced to the solution of the resolving equations. The automorphic system keeps the symmetry group of the original equations. Starting with a particular solution of the resolvent system, the automorphic system is fixed and all of its solutions can be obtained by applying symmetry transformations of the group to any particular solution of the automorphic system. This property makes it integrable. Thus non-invariant solutions can be obtained explicitly provided we can find solutions of the resolving equations. Depending on the problem at hand, this task can be equally difficult as the original problem, or even harder, but also it can happen that even a simple solution of the resolving equations can lead to highly non-trivial solutions of the original differential equation.

We feel that group splitting provides a regular framework for obtaining non-invariant solutions of partial differential equations. In the case of invariant solutions we are dealing with a submanifold that has dimension smaller than the general solution manifold. This leads to symmetry reduction of the original equations with the loss of all other solutions. However, in group splitting there is no symmetry reduction because we are concerned with the orbits of non-invariant solutions and these orbits are themselves invariant submanifolds of the general solution manifold with the same dimension as the original solution manifold. In our approach we shall refer to group splitting by the more descriptive name *group foliation* because we are foliating the solution manifold into invariant submanifolds of orbits of solutions.

An important new contribution to group foliation that we present here lies in the systematic use of the commutator algebra of operators of invariant differentiation. In this way we avoid the usual procedure of arriving at the resolving equations by a straightforward but lengthy derivation of integrability conditions of the automorphic system. The commutator algebra of the operators of invariant differentiation also results in important relations between higher-order differential invariants which provides a check on the completeness of the resolvent system. Finally, the commutator algebra together with the Jacobi identities turns out to be equivalent to the complete set of resolving equations and provides a commutator representation of the resolvent system. It seems to be an analogue of the Lax representation for the multi-dimensional case. This would make the resolvent system integrable if some generalization of the inverse scattering method can be associated with this commutator representation. Thus the use of the commutator algebra of operators of invariant differentiation greatly simplifies the theory of group foliation both in conceptual understanding as well as in calculations.

Classical symmetry reduction imposes extra conditions in the form of new partial differential equations compatible with the original equation in order to make the process of finding the invariant solution simpler than the general solution. A similar simplification without restriction to invariant solutions is provided by group foliation, namely, for each particular solution of the resolving equations the role of such extra conditions is played by the automorphic system. Just as in the case of symmetry reduction we now have a simpler problem of finding a particular solution instead of a general one. However, the important difference between these two approaches lies in the fact that this particular solution need not be an invariant solution of the original equation.

We choose for the group foliation the most essential infinite subgroup of the total symmetry group of CMA_2 obtained by Boyer and Winternitz [7], namely *the group of all biholomorphisms* *in two complex variables with unit Jacobian* which will be referred to as

$$\mathrm{BiHol}(M). \tag{4}$$

Then we construct the basis of differential invariants and operators of invariant differentiation for this group. All the differential invariants are obtained starting from the basis invariants. It turns out that this can be accomplished at the level of third prolongation. We restrict ourselves only to the set of invariants which survive on the solution manifold of CMA_2. Then we fix the general form of automorphic system by choosing a set of four basis differential invariants which we consider as independent variables and regard the remaining four to be arbitrary smooth functions of the first set. Finally, we derive the integrability conditions of the automorphic system in its general form to end up with the resolving equations. These are partial differential equations for the functions expressing the dependent differential invariants in terms of the independent ones. In principle one could find out which solutions of the resolving equations correspond to invariant solutions of CMA_2 and eliminate them from further consideration. However, here we shall not study such additional requirements. If any other solution of the resolving equations is obtained, then by virtue of the automorphic property the system can be integrated to yield non-invariant solutions of CMA_2.

The differential invariants are produced by the basis invariants and the action of the operators of invariant differentiation on the basis. Hence one important by-product of our analysis is that with this knowledge of all differential invariants we *construct the full set of partial differential equations of arbitrary order in one real unknown and two complex independent variables that remain invariant under the group of arbitrary biholomorphisms with unit Jacobian*. CMA_2 itself is expressed as the constancy of a second-order basis differential invariant.

3. Prolongation spaces, basis vector fields and differential invariants

The elliptic CMA_2 is given explicitly by

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = 1 \tag{5}$$

with the unknown real-valued function u depending on two complex coordinates ζ^1, ζ^2 and their complex conjugates $\bar{\zeta}^1, \bar{\zeta}^2$. We shall consider the prolongation structure of CMA_2 where the unknown u and its partial derivatives will be regarded as new independent variables. The order of prolongation, K, will refer to the maximal order of partial derivatives of u which are included as independent variables. The original complex manifold M will be prolonged to $Z^K(M)$ with partial derivatives up to order K acting as new local coordinates. We have the operator of total differentiation

$$D_i = \partial_i + \sum_I u_{Ii}\ \partial_{u_I} \tag{6}$$

where the symbol $I = (i_1, i_2, \ldots, i_s)$ is a multi-index and for each value of l we have $i_l = 1, 2, \bar{1}, \bar{2}$. The sum in equation (6) runs only over the values of a multi-index for which $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s$ where we introduce an ordering relation $1 < 2 < \bar{1} < \bar{2}$. This notation is standard for prolongation structure and in the theory of group analysis [2, 8–10]. The definition of the operator of total differentiation is such that when D_i is formally applied to any function depending on $\zeta^i, \bar{\zeta}^j, u, u_i, u_{i_1i_2}, \ldots, u_{i_1i_2\ldots i_s}$, with the partial derivatives $u_{i_1i_2\ldots i_s}$ treated as independent variables, they produce the same result as a common total partial derivative applied to a composite function obtained by the substitution for u a particular solution. That is, when we substitute for u a particular solution depending on the original independent variables, namely the coordinates, then on the solution manifold $u_{i_1 i_2 \dots i_s}$ represents the usual partial derivative.

The total Lie group of symmetry point transformations[1] for equation (5) was determined by Boyer and Winternitz [7] who also obtained the complete classification of its symmetry subgroups. For group foliation we shall use its most essential infinite subgroup, namely the group of all unimodular biholomorphisms. The generator of $\mathrm{BiHol}(M)$ is given by

$$X = \tfrac{1}{2}\mathrm{i}\left(U_2\partial_1 - U_1\partial_2 - U_{\bar{2}}\partial_{\bar{1}} + U_{\bar{1}}\partial_{\bar{2}}\right) \tag{7}$$

where ∂_i denotes the partial derivative with respect to ζ^i. U is a real-valued function of two complex variables subject to the linear system of equations

$$U_{1\bar{1}} = 0 \qquad U_{2\bar{2}} = 0 \qquad U_{1\bar{2}} = 0 \qquad U_{2\bar{1}} = 0 \tag{8}$$

with the solution that U is the real part of an arbitrary biholomorphic, or antibiholomorphic function. This result is responsible for the fact that only the purely holomorphic, or antiholomorphic partial derivatives of U will survive in the prolongation formulae below.

Differential invariants are invariants of the symmetry group in the prolongation space. Differential invariants up to a certain order will depend on the derivatives of unknowns up to that order. The maximal order of these derivatives is called the *order of the differential invariant.* We need to find all functionally independent differential invariants up to a certain order K for the infinite symmetry subgroup generated by the vector field (7). In other words, we shall find all independent partial differential equations up to order K with the one unknown and four independent variables which keep the same infinite symmetry subgroup as CMA_2.

It is important to calculate the number of functionally independent differential invariants for the fixed order K of a prolongation for the symmetry group used for the foliation. In what follows *number* refers to the number of real variables only and a complex variable will be counted as two. For determining the number of functionally independent invariants we must know the number r_K of *linearly unconnected generators* in the prolongation space[2] which gives us the dimension of generic orbits of the symmetry group [2]. By definition, linearly unconnected generators cannot be expressed as linear combinations of other generators with functional coefficients depending upon local coordinates in the prolongation space. This is a much stronger requirement than linear independence where only constant coefficients are considered. If K is the order of prolongation, the maximal order of partial derivatives of the unknown u involved, and ν_K is the dimension of the prolongation space, then the number of functionally independent differential invariants up to the order K with respect to the symmetry group is $\nu_K - r_K$. It is convenient to keep in mind the general expression for the dimension of the prolongation space [2]

$$\nu_K = p + q\,\frac{(K+p)!}{K!\,p!} \qquad K = 0, 1, 2, \dots$$

where p and q are the numbers of independent and dependent variables, respectively, and K is the order of prolongation. For equation (5) $p = 4$, $q = 1$ and we obtain

$$\nu_K = 4 + \frac{(K+4)!}{K!\,4!} \tag{9}$$

for the dimension of the prolongation space of order K for CMA_2. We have already remarked that by number we understand the number of real variables, this was already the case above when we took $p = 4$.

[1] It turns out that CMA_2 does not possess a symmetry group of contact transformations.

[2] As well as for the unprolonged space with $K = 0$.

We start from the unprolonged space for equation (5) with the local coordinates $\zeta^1, \zeta^2, \bar{\zeta}^1, \bar{\zeta}^2, u$ and the dimension $\nu_0 = 5$. The generator (7) of any one-parameter subgroup of the infinite symmetry group is

$$
\begin{aligned}
&X = U_1 X_1 + U_2 X_2 + U_{\bar{1}} X_{\bar{1}} + U_{\bar{2}} X_{\bar{2}} \\
&X_1 = -\tfrac{1}{2}\mathrm{i}\,\partial_2 \qquad X_2 = \tfrac{1}{2}\mathrm{i}\,\partial_1 \\
&X_{\bar{1}} = \tfrac{1}{2}\mathrm{i}\,\partial_{\bar{2}} \qquad X_{\bar{2}} = -\tfrac{1}{2}\mathrm{i}\,\partial_{\bar{1}}
\end{aligned}
\tag{10}
$$

which is the same as the linear combination of the four basis generators (7). Partial derivatives in the independent variables give rise to the four linearly unconnected generators X_i, so that $r_0 = 4$, $\nu_0 - r_0 = 1$ and we have only one obvious independent invariant, namely u, with respect to the infinite group generated by the vector fields X of the form (10).

This result means that there are no non-constant invariant solutions of CMA_2 with respect to the whole infinite symmetry subgroup $\mathrm{BiHol}(M)$. Thus whenever we mention invariant solutions we mean solutions of CMA_2 invariant with respect to linear combinations of generators of the following subgroups of the total symmetry group of CMA_2; namely, some one-parameter subgroups of $\mathrm{BiHol}(M)$ and other subgroups which are not involved in our construction such as the groups of translations, rotations and dilatations [7].

For the first prolongation we take the four first derivatives of u as additional local coordinates so that the dimension of the space becomes $\nu_1 = 9$. Using standard formulae [2, 8–10] we obtain the first prolongation of the symmetry generator (7) which results in additional terms to X proportional to all the second derivatives of U apart from those which vanish owing to equations (8)

$$
\overset{1}{X} = X + \sum_{ij} U_{ij} \overset{1}{X}_{ij} + \sum_{\bar{i}\bar{j}} U_{\bar{i}\bar{j}} \overset{1}{X}_{\bar{i}\bar{j}}
\tag{11}
$$

where

$$
\begin{aligned}
&\overset{1}{X}_{11} = \tfrac{1}{2}\mathrm{i}\,u_2 \partial_{u_1} \qquad \overset{1}{X}_{22} = -\tfrac{1}{2}\mathrm{i}\,u_1 \partial_{u_2} \\
&s\overset{1}{X}_{12} = \tfrac{1}{2}\mathrm{i}\,(u_2 \partial_{u_2} - u_1 \partial_{u_1})
\end{aligned}
\tag{12}
$$

and $\overset{1}{X}_{\bar{i}\bar{j}}$ is the complex conjugate of the corresponding $\overset{1}{X}_{ij}$. This is a general property at each level of prolongation so that here and in the following we do not need to write the barred quantities. At this level we have only four additional basis vector fields $\partial_{u_i}, \partial_{u_{\bar{i}}}$ which are obviously linearly unconnected. This results in two linear connections between six operators (12) and their complex conjugates reducing the total number to $r_1 = 8$. Hence the number of independent differential invariants is still equal to $\nu_1 - r_1 = 1$ and we are left again with the invariant u.

Next we consider the second prolongation with $K = 2$ and dimension $\nu_2 = 19$ where ten second derivatives of u come into play as the new local coordinates. Standard prolongation formulae [2, 8–10] for the generator (7) lead to

$$
\overset{2}{X} = X + \sum_{ij} U_{ij} \overset{2}{X}_{ij} + \sum_{\bar{i}\bar{j}} U_{\bar{i}\bar{j}} \overset{2}{X}_{\bar{i}\bar{j}} + \sum_{ijk} U_{ijk} \overset{2}{X}_{ijk} + \sum_{\bar{i}\bar{j}\bar{k}} U_{\bar{i}\bar{j}\bar{k}} \overset{2}{X}_{\bar{i}\bar{j}\bar{k}}
\tag{13}
$$

containing all the third purely holomorphic, or antiholomorphic derivatives of U which do not vanish as a consequence of equations (8). In equation (13) the operators $\overset{2}{X}_{ij}$ have additional terms compared with $\overset{1}{X}_{ij}$ of equation (12) that come from the second prolongation

$$
\begin{aligned}
\overset{2}{X}_{11} &= \tfrac{1}{2}\mathrm{i}\,(u_2\partial_{u_1} + 2u_{21}\partial_{u_{11}} + u_{22}\partial_{u_{21}} + u_{2\bar 1}\partial_{\bar 1 1} + u_{2\bar 2}\partial_{u_{\bar 2 1}})\\
\overset{2}{X}_{22} &= -\tfrac{1}{2}\mathrm{i}\,(u_1\partial_{u_2} + u_{11}\partial_{u_{12}} + 2u_{12}\partial_{u_{22}} + u_{1\bar 1}\partial_{\bar 1 2} + u_{1\bar 2}\partial_{u_{\bar 2 2}})\\
\overset{2}{X}_{12} &= \tfrac{1}{2}\mathrm{i}(u_2\partial_{u_2} - u_1\partial_{u_1} + 2u_{22}\partial_{u_{22}} - 2u_{11}\partial_{u_{11}} + u_{2\bar 1}\partial_{u_{\bar 1 2}} + u_{2\bar 2}\partial_{u_{\bar 2 2}} - u_{1\bar 1}\partial_{u_{\bar 1 1}} - u_{1\bar 2}\partial_{u_{\bar 2 1}})
\end{aligned}
\tag{14}
$$

and operators with three indices

$$
\begin{aligned}
\overset{2}{X}_{111} &= \tfrac{1}{2}\mathrm{i}\,u_2\partial_{u_{11}} & \overset{2}{X}_{112} &= \tfrac{1}{2}\mathrm{i}\,(u_2\partial_{u_{12}} - u_1\partial_{u_{11}})\\
\overset{2}{X}_{122} &= \tfrac{1}{2}\mathrm{i}\,(u_2\partial_{u_{22}} - u_1\partial_{u_{12}}) & \overset{2}{X}_{222} &= -\tfrac{1}{2}\mathrm{i}\,u_1\partial_{u_{22}}
\end{aligned}
\tag{15}
$$

which arise only in the second prolongation.

In the second prolongation all six operators $\overset{2}{X}_{ij}$ become linearly unconnected, while among the operators $\overset{2}{X}_{ijk}$, which appear as prolongations of the six unconnected operators $\overset{2}{X}_{ij}$, also six are linearly unconnected and there exist two linear relations connecting them. To obtain them explicitly we shall solve six out of the eight equations (15) and their complex conjugates with respect to each of the six partial derivative operators $\partial_{u_{ij}}$, $\partial_{u_{\bar i\bar j}}$ and substitute the result into the other two of these equations.

At the level of second prolongation we have $r_2 = 16$ linearly unconnected vector fields $\overset{2}{X}_i$, $\overset{2}{X}_{ij}$, $\overset{2}{X}_{ijk}$ and the number of functionally independent differential invariants is $\nu_2 - r_2 = 3$. Apart from the original invariant u we have two second-order invariants

$$\mu = u_{1\bar 1}u_{2\bar 2} - u_{1\bar 2}u_{2\bar 1} \tag{16}$$

$$\lambda = u_1u_{\bar 2}u_{2\bar 1} + u_2u_{\bar 1}u_{1\bar 2} - u_1u_{\bar 1}u_{2\bar 2} - u_2u_{\bar 2}u_{1\bar 1} \tag{17}$$

which consist of the left-hand side of CMA_2 and its Lagrangian [11]. Thus we are left with only two independent invariant variables on the solution manifold of CMA_2.

In the theory of group foliation it is generally assumed that the number of independent variables in the resolving equations must be the same as in the original equations [2] in order to ensure that no solutions of the original equations are lost. This means that for CMA_2 we need at least five independent differential invariants. Four of these differential invariants will be regarded as independent variables and the remaining ones will be considered to be a function of these four. At the level of second prolongation we have only two, so the third prolongation is necessary.

For the third prolongation $K = 3$ and the dimension is $\nu_3 = 39$ with 20 independent third partial derivatives of u acting as the new local coordinates. In this case prolongation formulae for the generator X given in equation (7) lead to the expression

$$
\begin{aligned}
\overset{3}{X} = X &+ \sum_{ij} U_{ij}\overset{3}{X}_{ij} + \sum_{\bar i\bar j} U_{\bar i\bar j}\overset{3}{X}_{\bar i\bar j} + \sum_{ijk} U_{ijk}\overset{3}{X}_{ijk} + \sum_{\bar i\bar j\bar k} U_{\bar i\bar j\bar k}\overset{3}{X}_{\bar i\bar j\bar k}\\
&+ \sum_{ijkl} U_{ijkl}\overset{3}{X}_{ijkl} + \sum_{\bar i\bar j\bar k\bar l} U_{\bar i\bar j\bar k\bar l}\overset{3}{X}_{\bar i\bar j\bar k\bar l}
\end{aligned}
\tag{18}
$$

containing terms proportional to all those fourth derivatives of U that do not vanish by equations (8). In equation (18) the operators $\overset{3}{X}_{ij}$, $\overset{3}{X}_{ijk}$ have additional terms as compared with $\overset{2}{X}_{ij}$, $\overset{2}{X}_{ijk}$ of equations (14) and (15) arising from the third prolongation.

To see that the third prolongation is sufficient for our purposes we need to evaluate the number of linearly unconnected vector fields. Indeed, we have four vector fields X_i and six $\overset{3}{X}_{ij}$ which were linearly unconnected in the second prolongation and also keep this property in the third prolongation. This brings their total number to ten. Furthermore, we expect that all eight vector fields $\overset{3}{X}_{ijk}$ which were connected in the second prolongation by two linear relations become linearly unconnected in the third prolongation, increasing the total number of such operators to 18. Finally, ten operators $\overset{3}{X}_{ijkl}$ arise only in the third prolongation as a result of prolongation of the eight linearly unconnected operators $\overset{3}{X}_{ijk}$. Hence we expect that only eight of them will be linearly unconnected so that two linear relations must exist among them. That adds up to the total number of $r_3 = 26$ linearly unconnected generators and to $\nu_3 - r_3 = 39 - 26 = 13$ functionally independent differential invariants at the third order of prolongation which at this stage appears to be sufficient for group foliation. However, the precise determination that third-order prolongation is sufficient for group foliation of CMA_2 requires knowledge of the third-order differential invariants. Then we can explicitly check which of these differential invariants become constant on the solution manifold of CMA_2 and thereby determine the number of truly independent variable invariants. For this purpose we shall need the operators of invariant differentiation and the full list of basis differential invariants which provide the complete set of differential invariants. This will be given in the next section.

The calculation of third-order invariants is a quite complicated problem and we shall delegate the details to the appendix. Here we shall present the final result of these calculations which consists of the list of all the functionally independent third-order differential invariants

$$l = u_{\bar2}(u_1\varphi_2 - u_2\varphi_1) - u_{\bar1}(u_1\psi_2 - u_2\psi_1) \tag{19}$$

$$L = u_{\bar2}(\bar\varphi\varphi_2 - \bar\psi\varphi_1) - u_{\bar1}(\bar\varphi\psi_2 - \bar\psi\psi_1) \tag{20}$$

$$m = u_{2\bar2}\varphi_1 - u_{1\bar2}\varphi_2 - u_{2\bar1}\psi_1 + u_{1\bar1}\psi_2 \tag{21}$$

$$M = \bar\varphi(u_{1\bar1}u_{22\bar2} - u_{2\bar1}u_{12\bar2} - u_{1\bar2}u_{22\bar1} + u_{2\bar2}u_{12\bar1}) - \bar\psi(u_{1\bar1}u_{12\bar2} - u_{2\bar1}u_{11\bar2} - u_{1\bar2}u_{12\bar1} + u_{2\bar2}u_{11\bar1}) \tag{22}$$

$$\nu = \psi(u_1\varphi_2 - u_2\varphi_1) - \varphi(u_1\psi_2 - u_2\psi_1) \tag{23}$$

which are all complex. Here φ and ψ are defined by

$$\varphi \overset{def}{=} u_1u_{2\bar1} - u_2u_{1\bar1} \qquad \psi \overset{def}{=} u_1u_{2\bar2} - u_2u_{1\bar2} \tag{24}$$

which turn out to be important, though non-invariant, quantities. Finally, for future purposes it will be convenient to note that the second-order differential invariants are expressible as

$$\mu = \bar\mu = \tfrac{1}{2}(\psi_{\bar1} - \varphi_{\bar2}) \qquad \lambda = \bar\lambda = u_{\bar2}\varphi - u_{\bar1}\psi \tag{25}$$

in terms of φ, ψ.

In the next section we shall show that some of these differential invariants can be obtained very simply by the action of the operators of invariant differentiation on second-order invariants.

Thus, at the third order of prolongation all the functionally independent differential invariants of the symmetry group of CMA_2, namely the group of arbitrary unimodular biholomorphisms, are given by arbitrary smooth functions of

$$u \quad \lambda \quad \mu \quad l \quad \bar l \quad L \quad \bar L \quad M \quad \bar M \quad m \quad \bar m \quad \nu \quad \bar\nu. \tag{26}$$

In section 5 we shall use the explicit expressions for the differential invariants in group foliation of CMA_2.

4. Operators of invariant differentiation and basis of differential invariants

Operators of invariant differentiation commute with the prolonged generator of any one-parameter subgroup of the symmetry group used for the foliation [2]. They are linear combinations of the operators of total differentiation with functional coefficients depending upon local coordinates of the prolongation space. In the case of CMA_2 there exist four such operators

$$\delta = u_1 D_2 - u_2 D_1 \tag{27}$$

and

$$\Delta = \bar{\varphi}\, D_2 - \bar{\psi}\, D_1 \tag{28}$$

together with their complex conjugates. The action of the operators of invariant differentiation increases the order of prolongation

$$\left.\begin{matrix}\delta\\ \Delta\end{matrix}\right\}: \quad Z^K(M) \to Z^{K+1}(M) \tag{29}$$

and acting on any differential invariant they produce again a differential invariant of one higher order. In C^2 there exist two independent holomorphic and antiholomorphic operators of differentiation and the total number of independent operators of invariant differentiation must necessarily be the same. The quantities φ, ψ defined in equations (24) are components of the operator of invariant differentiation $\bar{\Delta}$. These are not differential invariants themselves but they play an important role in the construction of differential invariants. Finally, we note the results

$$\varphi = \delta(u_{\bar{1}}) \qquad \psi = \delta(u_{\bar{2}}) \qquad \Delta(\ln u_{\bar{k}}) = -\mu \qquad k = 1, 2 \tag{30}$$

for the action of the operators of invariant differentiation on the first partial derivatives of u.

With the aid of operators of invariant differentiation it is possible to present some third-order independent differential invariants in compact form. We have 13 differential invariants, namely u at zeroth order, two second-order invariants λ, μ which are real and ten third-order invariants l, L, m, M, ν together with their complex conjugates.

The second-order real invariant

$$\lambda = \Delta(u) \tag{31}$$

is quasilinear in the second derivatives of u. The operator of invariant differentiation δ annihilates u

$$\delta(u) = 0 \tag{32}$$

and does not give rise to a new differential invariant, u is in the kernel of δ. The invariant μ, which is just the left-hand side of CMA_2, is a basis invariant. On the solution manifold of CMA_2 we have $\mu = 1$. In order to account for CMA_2 in section 5 it will be useful to express μ through φ and ψ from equation (25).

Eight third-order differential invariants are obtained by the operators of invariant differentiation acting on two second-order invariants λ and μ

$$l = \delta(\lambda) \tag{33}$$

$$L = \Delta(\lambda) + \mu\lambda \tag{34}$$

$$m = \delta(\mu) \tag{35}$$

$$M = \Delta(\mu) \tag{36}$$

together with their complex conjugates. The remaining two third-order invariants ν and $\bar{\nu}$ are basis invariants which cannot be produced by the action of operators of invariant differentiation on second-order invariants. This was the reason for the necessity of the third prolongation. Nevertheless, the basis invariant ν can be expressed through the operator δ acting on the non-invariant quantities φ and ψ

$$\nu = \psi\delta(\varphi) - \varphi\delta(\psi) \tag{37}$$

which is the same as equation (23). All of these invariants were obtained by a straightforward solution of the equations expressing the invariance conditions for the third prolongation. Here we see that apart from the basis invariant ν third-order differential invariants can be obtained by the action of the operators of invariant differentiation on second-order invariants.

Operators of invariant differentiation form the algebra of vector fields determined by the commutation relations

$$[\delta, \bar{\delta}] = \bar{\Delta} - \Delta \qquad [\Delta, \bar{\Delta}] = \frac{1}{\lambda}\,(L\bar{\Delta} - \bar{L}\Delta) + \left(\frac{\mu l}{\lambda} - m\right)\bar{\delta} - \left(\frac{\mu\bar{l}}{\lambda} - \bar{m}\right)\delta \tag{38}$$

$$[\delta, \Delta] = \frac{1}{\lambda}\left(l\Delta - (L + \lambda\mu)\delta\right) \qquad [\delta, \bar{\Delta}] = \mu\delta - \frac{\nu}{\lambda}\,\bar{\delta} + \frac{l}{\lambda}\,\bar{\Delta} \tag{39}$$

and the complex conjugates of equations (39). Due to these relations not all of the higher-order invariants obtained by the action of the operators of invariant differentiation are functionally independent. By applying six commutation relations (38) and (39) including their complex conjugates to the invariants λ and μ we derive 12 relations between fourth-order invariants generated by the action of the operators of invariant differentiation on the third-order invariants

$$\delta(\bar{l}) = \bar{\delta}(l) + \bar{L} - L \tag{40}$$

$$\Delta(l) = \delta(L) + \mu l - \lambda m \tag{41}$$

$$\Delta(\bar{L}) = \bar{\Delta}(L) + \lambda(M - \bar{M}) + \bar{m}l - m\bar{l} \tag{42}$$

$$\delta(\bar{L}) = \bar{\Delta}(l) + \mu l + \lambda m + \frac{1}{\lambda}\,(l\bar{L} - \nu\bar{l}) \tag{43}$$

$$\delta(M) = \Delta(m) + \frac{1}{\lambda}\left(lM - (L + \lambda\mu)m\right) \tag{44}$$

$$\delta(\bar{m}) = \bar{\delta}(m) + \bar{M} - M \tag{45}$$

$$\Delta(\bar{M}) = \bar{\Delta}(M) + \frac{1}{\lambda}\,(L\bar{M} - \bar{L}M) + \frac{\mu}{\lambda}\,(l\bar{m} - \bar{l}m) \tag{46}$$

$$\delta(\bar{M}) = \bar{\Delta}(m) + \mu m + \frac{1}{\lambda}\,(l\bar{M} - \nu\bar{m}) \tag{47}$$

and their complex conjugates. In the next section we show that an additional five relations follow from the explicit expressions for differential invariants by cross differentiation and combining them with CMA_2.

To summarize, the basis of differential invariants consists of

$$u \quad \mu \quad \nu \quad \bar{\nu} \tag{48}$$

and the full set of differential invariants for CMA_2 *at arbitrary order* can be obtained by the repeated action of the operators of invariant differentiation on basis invariants and taking arbitrary smooth functions of the result. The crucial point is that no further *basis* invariants appear at higher orders of prolongation. This can be surmised by counting the number of

Table 1. Differential invariants. The arrows $\uparrow$ and $\nearrow$ represent the action of the operators of invariant differentiation Δ and δ, respectively.

Z^4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
	$\uparrow$	$\nearrow$	$\uparrow$	$\nearrow$	$\uparrow$	$\nearrow$	$\uparrow$	$\nearrow$	$\uparrow$	$\nearrow$
Z^3	$L-\lambda\mu$		l		M		m		ν	
	$\uparrow$	$\nearrow$			$\uparrow$	$\nearrow$				
Z^2	λ		0		μ					
	$\uparrow$	$\nearrow$								
Z^0	u									

functionally independent invariants expected at order $K > 3$ and the number we would obtain by applying the operators of invariant differentiation to invariants at order $K - 1$. Three is the last level at which the expected number of invariants is more than the number of invariants we can construct by applying operators of invariant differentiation. The situation changes dramatically at level $K = 4$ where the number of fourth-order differential invariants obtained by applying the operators of invariant differentiation is 40 and on account of 12 relations between them and five additional relations given in the next section we have 23 invariants. This coincides with the number of fourth-order independent invariants expected at the fourth prolongation. Hence we have the constructive proof that we have obtained the full set of basis differential invariants.

In table 1 we show the basis differential invariants and further differential invariants obtained by the action of the operators of invariant differentiation. This knowledge of all differential invariants enables us construct the full set of independent partial differential equations of arbitrary order in one real unknown and two complex independent variables that remain invariant under the group of arbitrary unimodular biholomorphisms. In particular, CMA_2 is given by $\mu = 1$.

5. Automorphic system and resolving equations

Up to now we have not used CMA_2 itself but only its symmetry properties. To account for the equation itself we consider the differential invariants (26) on a solution manifold of this equation. Then five of them are eliminated,

$$\mu = 1 \qquad M = 0 \qquad m = 0 \qquad \bar{M} = 0 \qquad \bar{m} = 0 \tag{49}$$

and we are left with the eight invariants

$$u \quad \lambda \quad l \quad \bar{l} \quad L \quad \bar{L} \quad \nu \quad \bar{\nu}. \tag{50}$$

For group foliation we must choose among these eight a set of four which will be considered as independent variables. In the theory of group foliation it is customary [2] that in order not to lose any solutions the number of differential invariants to be regarded as independent has to be the same as the number of independent variables in the original equation (5), namely four.

We choose as independent variables u, the second-order invariant λ and the simplest pair $l, \bar{l}$ of third-order invariants. Then the remaining four invariants $L, \bar{L}, \nu, \bar{\nu}$ will be smooth functions $F, \bar{F}, G, \bar{G}$ of the independent ones. It should be noted that this choice of independent variables is by no means unique and there are other possible choices of independent variables, however, then the resulting relations are much more complicated. Thus we *fix* the general form of the *automorphic system* for CMA_2

$$\begin{aligned} L &= F(u, \lambda, l, \bar{l}) \\ \nu &= G(u, \lambda, l, \bar{l}) \end{aligned} \tag{51}$$

together with their complex conjugate equations. On account of equations (51) and (49) the relations (40)–(47) between the fourth-order differential invariants become

$$\delta(\bar{l}) = \bar{\delta}(l) + \bar{F} - F \tag{52}$$

$$\bar{F}_l \Delta(l) + \bar{F}_{\bar{l}} \Delta(\bar{l}) + \lambda \bar{F}_u + (F - \lambda)\bar{F}_\lambda = F_l \bar{\Delta}(l) + F_{\bar{l}} \bar{\Delta}(\bar{l}) + \lambda F_u + (\bar{F} - \lambda) F_\lambda \tag{53}$$

$$\Delta(l) = F_l \delta(l) + F_{\bar{l}} \delta(\bar{l}) + l(F_\lambda + 1) \tag{54}$$

$$\bar{\Delta}(l) = \bar{F}_l \delta(l) + \bar{F}_{\bar{l}} \delta(\bar{l}) + l(\bar{F}_\lambda - 1) - \frac{1}{\lambda}(l\bar{F} - \bar{l}G) \tag{55}$$

and their complex conjugates with all other relations identically satisfied due to equations (49). The unknown functions F, G are complex and they will satisfy partial differential equations resulting from algebraic compatibility and integrability conditions of the set of relations (52)–(55) completed by five more independent relations. These additional equations are also projections of some further relations between fourth-order invariants which follow from integrability conditions for the partial differential equations (51) between themselves as well as with CMA_2. The character of equations (51) as a set of partial differential equations comes from the explicit expressions for the differential invariants obtained earlier. The resulting partial differential equations for the unknown functions F, G are called resolving equations. Our main goal in this section is to obtain this resolvent system.

There are two types of integrability conditions arising from the automorphic system (51) together with CMA_2. First, there exist *inner integrability conditions* for equations (51) alone. They arise from the explicit expressions for the differential invariants and equations (51). Then there is the problem of compatibility of equations (51) with CMA_2 itself which will give rise to *outer integrability conditions*.

To study the *inner integrability conditions* of the automorphic system (51) we rewrite the explicit expressions (19), (21), (20), (23) for the differential invariants l, m, L, ν, respectively, using the operators of invariant differentiation. As an example, we note that, in particular, m is expressible as

$$m = \frac{1}{\lambda}\left[\mu\big(u_{\bar{2}}\delta(\varphi) - u_{\bar{1}}\delta(\psi)\big) + \psi\Delta(\varphi) - \varphi\Delta(\psi)\right].$$

Then using the automorphic system (51) and the expressions for differential invariants we obtain a linear algebraic system for the operators of invariant differentiation acting on φ and ψ,

$$\begin{aligned} u_{\bar{2}}\delta(\varphi) - u_{\bar{1}}\delta(\psi) &= l \\ \varphi\Delta(\psi) - \psi\Delta(\varphi) &= \mu l - \lambda m \\ u_{\bar{2}}\Delta(\varphi) - u_{\bar{1}}\Delta(\psi) &= F(u, \lambda, l, \bar{l}) \\ \psi\delta(\varphi) - \varphi\delta(\psi) &= G(u, \lambda, l, \bar{l}) \end{aligned} \tag{56}$$

which has a non-vanishing determinant provided $\lambda \neq 0$. In fact, a detailed analysis of the condition $\lambda = \text{constant}$ shows that it is incompatible with CMA_2 itself. Hence we take $\lambda \neq 0$ and solve equations (56) algebraically,

$$\delta(\varphi) = \frac{1}{\lambda}(\varphi l - u_{\bar{1}}G) \qquad \delta(\psi) = \frac{1}{\lambda}(\psi l - u_{\bar{2}}G) \tag{57}$$

$$\Delta(\varphi) = \frac{1}{\lambda}\big(\varphi F + u_{\bar{1}}(\mu l - \lambda m)\big) \qquad \Delta(\psi) = \frac{1}{\lambda}\big(\psi F + u_{\bar{2}}(\mu l - \lambda m)\big) \tag{58}$$

which brings them to a form suitable for investigating their integrability conditions upon cross differentiation with δ and Δ. The result

$$G_l \Delta(l) + G_{\bar{l}} \Delta(\bar{l}) + \delta(l) = G - \lambda G_u - (F - \lambda)G_\lambda + \frac{3}{\lambda}(FG + l^2) \tag{59}$$

is expressible solely through differential invariants and operators of invariant differentiation. However, this equation cannot be regarded as a resolving equation because it contains fourth-order differential invariants generated by the action of operators of invariant differentiation δ and Δ on the third-order invariants $l, \bar{l}$. We note that if we substitute L and ν for F and G in equations (57) and (58), respectively, then cross differentiation results in a new complex relation expressing $\Delta(\nu)$ in terms of other fourth-order invariants generated by the operators of invariant differentiation acting on third-order invariants

$$\Delta(\nu) = \lambda\delta(m) - \mu\delta(l) + \mu\nu + \frac{3}{\lambda}\left(L\nu + \mu l^2 - \lambda l m\right) \tag{60}$$

and we can recognize this to be the same as equation (59) if we use equations (51) and account for CMA_2 by putting $\mu = 1$ and $m = 0$, cf equations (49).

Next we turn to the derivation of differential *outer integrability conditions* of the automorphic system (51) using CMA_2. For this purpose it will be convenient to start with the first derivatives of CMA_2 in the form (25)

$$(\bar{\psi}_{\bar{1}})_1 - (\bar{\varphi}_{\bar{1}})_2 = 0 \qquad (\bar{\psi}_{\bar{2}})_1 - (\bar{\varphi}_{\bar{2}})_2 = 0 \tag{61}$$

and combine these equations with the complex conjugate of the system (57) and (58). The resulting equations

$$\bar{F}_l\Delta(l) + \bar{F}_{\bar{l}}\Delta(\bar{l}) + \delta(\bar{l}) = -(F-\lambda)\bar{F}_\lambda - \lambda\bar{F}_u + \bar{F} + \frac{1}{\lambda}\left(|G|^2 + 2|F|^2 + 3|l|^2\right) \tag{62}$$

$$\Delta(\bar{l}) - \bar{G}_l\delta(l) - \bar{G}_{\bar{l}}\delta(\bar{l}) = l\bar{G}_\lambda + \bar{l} + \frac{1}{\lambda}(\bar{l}F - l\bar{G}) \tag{63}$$

are once again expressible solely through invariants. We remark again that if we substitute L and ν for F and G, respectively, in the complex conjugates of equations (57) and (58), then combining these equations with equations (61) we obtain two more complex equations:

$$\Delta(\bar{L}) + \mu\delta(\bar{l}) - \lambda\delta(\bar{m}) = \frac{1}{\lambda}\left(|\nu|^2 + 2|L|^2 + 3\mu|l|^2\right) + \mu\bar{L} - l\bar{m} - 2(m\bar{l} + \lambda\bar{M}) \tag{64}$$

$$\Delta(\bar{l}) - \delta(\bar{\nu}) = \frac{1}{\lambda}(\bar{l}L - l\bar{\nu}) + \mu\bar{l} - 3\lambda\bar{m} \tag{65}$$

which, as in equation (60), are relations between fourth-order invariants generated by the operators of invariant differentiation acting on third-order invariants. By using the relations (40) and (41) between fourth-order invariants we can check that equation (64) coincides with its complex conjugate. Hence in addition to 12 relations between these invariants obtained in the previous section we have now obtained five more relations (60), (64), (65) and their complex conjugates. Thus we obtain 23 independent fourth-order invariants generated by the operators of invariant differentiation which is the same as the number expected at the fourth prolongation. This completes the constructive proof that we have indeed obtained the complete set of basis differential invariants.

Next we turn to the derivation of resolving equations for the complex unknowns F and G. Up to now we have obtained 11 independent relations (52)–(55), (59), (62), (63) and their complex conjugates which are not yet the resolving equations since they involve actions of δ, Δ on $l, \bar{l}$. To get rid of such terms we choose these quantities as new unknowns, called *auxiliary variables*, and add new equations which follow from these definitions.

A straightforward procedure for deriving resolving equations is to solve algebraically the 11 relations mentioned above with respect to $\delta(l)$, $\delta(\bar{l})$, $\Delta(l)$ and $\Delta(\bar{l})$ and their complex conjugates to obtain three algebraic compatibility conditions implied by these relations. Next, applying cross differentiation by δ, Δ, $\bar{\delta}$, $\bar{\Delta}$ and using their commutation relations (38) and

(39), we obtain six integrability conditions of the equations, expressing $\delta(l)$, $\delta(\bar{l})$, $\Delta(l)$ and $\Delta(\bar{l})$ and their complex conjugates. Finally, we use the formulae for $\delta(l)$, $\delta(\bar{l})$, $\Delta(l)$ and $\Delta(\bar{l})$ and express them in terms of the auxiliary variables. In this way we obtain the complete set of 12 resolving equations. This approach enables us to avoid the usual lengthy procedure of cross differentiation using partial derivatives that has been commonly employed in group splitting. However, it also requires long calculations even though they are much simpler than the usual procedure. We shall therefore not follow this approach here.

The elegant way of deriving the resolving equations and presenting them in a compact form is to first simplify the commutator algebra of the operators of invariant differentiation by introducing a new pair of such operators Y and Z and their complex conjugates through the relations

$$\delta = \lambda Y \qquad \Delta = \lambda Z \qquad \bar{\delta} = \lambda \bar{Y} \qquad \bar{\Delta} = \lambda \bar{Z}. \tag{66}$$

Then the commutation relations (38) and (39) are simplified to

$$[Y, Z] = -\frac{2}{\lambda} Y \tag{67}$$

$$[Y, \bar{Z}] = \frac{1}{\lambda^2} (\bar{F} Y - G \bar{Y}) \tag{68}$$

$$[Y, \bar{Y}] = \frac{1}{\lambda^2} (\bar{l} Y - l \bar{Y}) + \frac{1}{\lambda} (\bar{Z} - Z) \tag{69}$$

$$[Z, \bar{Z}] = \frac{1}{\lambda^2} (l \bar{Y} - \bar{l} Y) + \frac{1}{\lambda} (\bar{Z} - Z). \tag{70}$$

We note that the commutator algebra of the original operators of invariant differentiation in equations (38) and (39) was given without projection onto the solution manifold of CMA_2, whereas here we have used equations (49) accounting for CMA_2 and the general form of the automorphic system (51). The action of the new operators of invariant differentiation on differential invariants is determined by the properties (31)–(34) combined with the definitions (66)

$$Y(u) = 0 \qquad Y(\lambda) = \frac{l}{\lambda} \qquad Z(u) = 1 \qquad Z(\lambda) = \frac{\Phi}{\lambda} \tag{71}$$

where $\Phi \stackrel{def}{=} F - \lambda$. The differential invariants (35) and (36) are annihilated on the solution manifold of CMA_2.

Now we introduce complex auxiliary variables $\vartheta, \sigma, \rho, \tau$ through the action of Y, Z, $\bar{Y}$ and $\bar{Z}$ on l and $\bar{l}$

$$Y(l) = \sigma \qquad Y(\bar{l}) = \tau \qquad Z(l) = \vartheta \qquad Z(\bar{l}) = \rho \tag{72}$$

and their complex conjugates which are all functions of the independent variables $u, \lambda, l, \bar{l}$. We shall see that the auxiliary variables will serve as additional unknowns in the first-order version of resolving equations. Formulae (71) and (72) determine the explicit expression for the new operators of invariant differentiation

$$\begin{aligned} Y &= \sigma \partial_l + \tau \partial_{\bar{l}} + \frac{l}{\lambda} \partial_\lambda \\ Z &= \vartheta \partial_l + \rho \partial_{\bar{l}} + \left(\frac{F}{\lambda} - 1\right) \partial_\lambda + \partial_u \end{aligned} \tag{73}$$

which consist of basis vector fields in the space of those differential invariants that are taken as independent variables with coefficients that are differential invariants themselves. This is the projection of the new operators of invariant differentiation onto the space of differential invariants.

A new and easy way of obtaining resolving equations is to apply the commutation relations (67)–(70) for $Y, Z, \bar{Y}, \bar{Z}$ to the differential invariants $u, \lambda, l, \bar{l}$ treated as independent variables. Applying them to u gives only identities. The resulting 12 resolving equations have the form

$$\bar{\tau} = \tau + \frac{1}{\lambda}(F - \bar{F}) \tag{74}$$

$$Y(\vartheta) - Z(\sigma) + \frac{2}{\lambda}\sigma = 0 \tag{75}$$

$$Y(\rho) - Z(\tau) + \frac{2}{\lambda}\tau = 0 \tag{76}$$

$$Z(\bar{\tau}) - \bar{Y}(\vartheta) + \frac{1}{\lambda^2}(F\bar{\tau} - \bar{G}\sigma) = 0 \tag{77}$$

$$\bar{Z}(\sigma) - Y(\bar{\rho}) + \frac{1}{\lambda^2}(\bar{F}\sigma - G\bar{\tau}) = 0 \tag{78}$$

$$\bar{Y}(\sigma) - Y(\bar{\tau}) + \frac{1}{\lambda^2}(\bar{l}\sigma - l\bar{\tau}) + \frac{1}{\lambda}(\bar{\rho} - \vartheta) = 0 \tag{79}$$

$$Z(\bar{\rho}) - \bar{Z}(\vartheta) - \frac{1}{\lambda}(\bar{\rho} - \vartheta) + \frac{1}{\lambda^2}(\bar{l}\sigma - l\bar{\tau}) = 0 \tag{80}$$

$$Y(F) = \vartheta - \frac{l}{\lambda} \tag{81}$$

$$Y(\bar{F}) = \bar{\rho} + \frac{l}{\lambda} + \frac{1}{\lambda^2}(l\bar{F} - \bar{l}G) \tag{82}$$

$$Z(\bar{F}) = -\tau + \frac{1}{\lambda}\bar{F} + \frac{1}{\lambda^2}(|G|^2 + 2|F|^2 + 3|l|^2) \tag{83}$$

$$Y(\bar{G}) = \rho - \frac{\bar{l}}{\lambda} - \frac{1}{\lambda^2}(\bar{l}F - l\bar{G}) \tag{84}$$

$$Z(G) = -\sigma + \frac{1}{\lambda}G + \frac{3}{\lambda^2}(l^2 + FG) \tag{85}$$

together with all of their complex conjugates. Not all of these equations are obtained in this manner[3]. Namely, equations (83)–(85) follow from equations (62), (63) and (59), respectively. The reason for the singular status of these three equations will be clear from the proof of the following theorem.

Theorem 1. *The commutator algebra* (67)–(70) *of the operators of invariant differentiation* $Y, Z, \bar{Y}, \bar{Z}$ *together with the Jacobi identities for these commutators is equivalent to the resolvent system and hence provides a commutator representation for this system.*

This theorem means that the complete set of the resolving equations is encoded in the commutator algebra of the operators of invariant differentiation.

For the proof we use the expressions (73) for the operators Y and Z and their complex conjugates in the commutation relations (67)–(70) to obtain immediately the resolving equations (74)–(82). To find the remaining three equations we impose the Jacobi identities

$$[[Y, Z], \bar{Y}] + [[Z, \bar{Y}], Y] + [[\bar{Y}, Y], Z] = 0 \tag{86}$$

[3] The arithmetics $4 \times 3 = 12$ is misleading since not all of the 12 equations obtained in this way are independent.

which entails the resolving equation (84) and

$$[[Y, Z], \bar{Z}] + [[Z, \bar{Z}], Y] + [[\bar{Z}, Y], Z] = 0 \tag{87}$$

which implies the remaining resolving equations (83) and (85). All other Jacobi identities are satisfied automatically by virtue of the resolving equations.

Thus we have the commutator operator representation for the four-dimensional resolvent system which seems to be the appropriate analogue of the Lax pair for the multi-dimensional case that includes the Jacobi identities for the commutators. The linear spectral problem for the set of operators of invariant differentiation should be studied in an attempt to arrive at a generalization of the inverse scattering method suitable for a multi-dimensional resolvent system. Then the complete system of resolving equations would present a very complicated example of a multi-dimensional integrable system which will make it interesting by itself even apart from its relation to CMA_2. Here we only note that, irrespective of its possible relation to the inverse scattering method this commutator representation can lead in a natural way to non-obvious ansätze for special classes of solutions of the resolvent system.

Next we consider the integrability conditions between different pairs of resolving equations chosen among the group of equation (81) and the complex conjugates of equations (82), (83) and also the pair of equations (85) and the complex conjugate of equation (84). For example, applying $\bar{Z}$ and Y to both sides of the first and the third of these equations, respectively, and then subtracting we find

$$\begin{aligned} Y(\tau) + \bar{Z}(\theta) - \frac{1}{\lambda^2}\,\bar{G}Y(G) = &-\frac{2}{\lambda^4}\big[l(|F|^2 + |G|^2 + 3|l|^2 - \lambda^2) + \bar{l}FG\big] \\ &+\frac{1}{\lambda^2}\big[2(\bar{\rho}F + \rho G) + \theta\bar{F} + 2\lambda\bar{\rho} + 3(\bar{l}\sigma + l\tau)\big] \\ &+\frac{1}{\lambda^3}\big[2l(F - \bar{F}) - \bar{l}G\big] \end{aligned} \tag{88}$$

which is a differential consequence of the resolving equations. Integrability conditions for all other pairs of the resolving equations mentioned above turn out to follow from equation (88) and equations of the original resolvent system.

It will be useful to present the resolving equations in explicit form using the definitions of the vector fields Y and Z:

$$\bar{\tau} = \tau + \frac{1}{\lambda}\,(F - \bar{F}) \tag{89}$$

$$\bar{\rho} = \bar{F}_l\sigma + \bar{F}_{\bar{l}}\tau + \frac{l}{\lambda}\,(\bar{F}_\lambda - 1) - \frac{1}{\lambda^2}(l\bar{F} - \bar{l}G) \tag{90}$$

$$\lambda(\sigma\vartheta_l + \tau\vartheta_{\bar{l}} - \vartheta\sigma_l - \rho\sigma_{\bar{l}} - \sigma_u) + l\vartheta_\lambda - \Phi\sigma_\lambda + 2\sigma = 0 \tag{91}$$

$$\lambda(\sigma\rho_l + \tau\rho_{\bar{l}} - \vartheta\tau_l - \rho\tau_{\bar{l}} - \tau_u) + l\rho_\lambda - \Phi\tau_\lambda + 2\tau = 0 \tag{92}$$

$$\lambda(\vartheta\bar{\tau}_l + \rho\bar{\tau}_{\bar{l}} - \bar{\tau}\vartheta_l - \bar{\sigma}\vartheta_{\bar{l}} + \bar{\tau}_u) + \Phi\bar{\tau}_\lambda - \bar{l}\vartheta_\lambda + \frac{1}{\lambda}\,(F\bar{\tau} - \bar{G}\sigma) = 0 \tag{93}$$

$$\lambda(\bar{\rho}\sigma_l + \bar{\vartheta}\sigma_{\bar{l}} - \sigma\bar{\rho}_l - \tau\bar{\rho}_{\bar{l}} + \sigma_u) + \bar{\Phi}\sigma_\lambda - l\bar{\rho}_\lambda + \frac{1}{\lambda}\,(\bar{F}\sigma - G\bar{\tau}) = 0 \tag{94}$$

$$\lambda(\bar{\tau}\sigma_l + \bar{\sigma}\sigma_{\bar{l}} - \sigma\bar{\tau}_l - \tau\bar{\tau}_{\bar{l}}) + \bar{l}\sigma_\lambda - l\bar{\tau}_\lambda + \frac{1}{\lambda}\,(\bar{l}\sigma - l\bar{\tau}) + \bar{\rho} - \vartheta = 0 \tag{95}$$

$$\lambda(\vartheta\bar{\rho}_l + \rho\bar{\rho}_{\bar{l}} - \bar{\rho}\vartheta_l - \bar{\vartheta}\vartheta_{\bar{l}} + \bar{\rho}_u - \vartheta_u) + \Phi\bar{\rho}_\lambda - \bar{\Phi}\vartheta_\lambda + \frac{1}{\lambda}\,(\bar{l}\sigma - l\bar{\tau}) - (\bar{\rho} - \vartheta) = 0 \tag{96}$$

$$l\bar{G}_\lambda + \bar{l} + (\bar{l}F - l\bar{G})/\lambda = \lambda(\rho - \bar{G}_l\sigma - \bar{G}_{\bar{l}}\tau) \tag{97}$$

$$l(1 + F_\lambda) = \lambda(\vartheta - F_l\sigma - F_{\bar{l}}\tau) \tag{98}$$

$$\lambda\bar{F}_u + (F - \lambda)\bar{F}_\lambda - \bar{F} - (|G|^2 + 2|F|^2 + 3|l|^2)/\lambda = -\lambda(\tau + \bar{F}_l\vartheta + \bar{F}_{\bar{l}}\rho) \tag{99}$$

$$\lambda G_u + (F - \lambda)G_\lambda - G - 3(FG + l^2)/\lambda = -\lambda(\sigma + G_l\vartheta + G_{\bar{l}}\rho). \tag{100}$$

This is the full resolvent system for CMA_2.

We shall conclude this section with the remark that the number of the resolving equations could be reduced if we directly substitute explicit expressions for the auxiliary variables σ, τ, ϑ and ρ in terms of F, G and their first derivatives. However, then we would end up with a much more complicated system of eight resolving equations containing second derivatives. The price that we pay for dealing with the system of first-order equations is the usual one of increasing the number of equations and unknowns. In addition to the original functions F, G we have 'gauge' degrees of freedom σ, τ, ϑ, ρ.

6. Conclusion

We have presented the general framework for finding non-invariant solutions of CMA_2 based on the method of group foliation. We have made an important development in the method of group foliation by showing the central role played by the commutator algebra of operators of invariant differentiation. In particular, we have shown that this algebra provides the commutator representation of the complete resolvent system. We emphasize that this method is a general one, applicable to all partial differential equations admitting an infinite Lie group of symmetries.

Studying the example of CMA_2 in detail we have obtained results which are by themselves major. We found the basis differential invariants from which all differential invariants can be obtained by the action of the operators of invariant differentiation. Consequently, we are able to write down all independent partial differential equations with one real unknown and two complex independent variables which keep the same infinite symmetry subgroup as CMA_2, namely the group of unimodular biholomorphisms.

For group foliation of CMA_2 we have established that third prolongation is sufficient. Then we set up the automorphic and the resolvent systems. We have used the operators of invariant differentiation and their commutator algebra as the basic objects in the theory of group foliation of CMA_2. They provide the simplest way of obtaining invariant integrability conditions of the automorphic system. The commutator algebra of operators of invariant differentiation produces relations between fourth-order invariants and integrability and algebraic compatibility conditions of these equations projected on to the solution manifold of CMA_2 are the resolving equations. More than that, this commutator algebra together with its Jacobi identities is equivalent to the complete resolvent system. Therefore, it provides a commutator representation of the resolvent system which is a generalization of the Lax representation for the multi-dimensional case. This representation may lead in a simple and natural way to reasonable anzätse for interesting classes of solutions of the resolving equations. A study of the linear spectral problem for the new operators of invariant differentiation Y, Z, $\bar{Y}$, $\bar{Z}$ will be important to decide whether the resolvent system is integrable within the framework of an appropriate generalization of the inverse scattering method.

In this paper we have not attempted to find non-invariant solutions, but rather concentrated on setting up the basic framework necessary for this purpose. We shall consider the problem of constructing non-invariant solutions of CMA_2 in a future publication.

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Appendix

Here we present some details of calculations of the third-order differential invariants.

The complete expressions for the vector fields $\overset{3}{X}_{ij}$ and $\overset{3}{X}_{ijk}$ introduced in equation (18) of section 3 are quite complicated. For example, the vector fields with two indices are given by

$$
\begin{aligned}
\overset{3}{X}_{11} = {} & \tfrac{1}{2}\mathrm{i}\,(u_2\partial_{u_1} + 2u_{21}\partial_{u_{11}} + u_{22}\partial_{u_{21}} + u_{2\bar{1}}\partial_{\bar{1}1} + u_{2\bar{2}}\partial_{u_{\bar{2}1}} + 2u_{21\bar{1}}\partial_{u_{\bar{1}11}} + 2u_{21\bar{2}}\partial_{u_{\bar{2}11}} + u_{2\bar{1}\bar{1}}\partial_{u_{\bar{1}\bar{1}1}} \\
& + u_{2\bar{1}\bar{2}}\partial_{u_{\bar{2}\bar{1}1}} + u_{22\bar{1}}\partial_{u_{\bar{1}21}} + u_{22\bar{2}}\partial_{u_{\bar{2}21}} + u_{2\bar{2}\bar{2}}\partial_{u_{\bar{2}\bar{2}1}}) \\
\overset{3}{X}_{22} = {} & -\tfrac{1}{2}\mathrm{i}\,(u_1\partial_{u_2} + u_{11}\partial_{u_{12}} + 2u_{12}\partial_{u_{22}} + u_{1\bar{1}}\partial_{\bar{1}2} + u_{1\bar{2}}\partial_{u_{\bar{2}2}} + u_{11\bar{1}}\partial_{u_{\bar{1}12}} + u_{11\bar{2}}\partial_{u_{\bar{2}12}} + u_{1\bar{1}\bar{1}}\partial_{u_{\bar{1}\bar{1}2}} \\
& + u_{1\bar{1}\bar{2}}\partial_{u_{\bar{2}\bar{1}2}} + 2u_{12\bar{1}}\partial_{u_{\bar{1}22}} + 2u_{12\bar{2}}\partial_{u_{\bar{2}22}} + u_{1\bar{2}\bar{2}}\partial_{u_{1\bar{2}\bar{2}}}) \qquad (101) \\
\overset{3}{X}_{12} = {} & \tfrac{1}{2}\mathrm{i}\big[u_2\partial_{u_2} - u_1\partial_{u_1} + 2(u_{22}\partial_{u_{22}} - u_{11}\partial_{u_{11}}) + u_{2\bar{1}}\partial_{u_{\bar{1}2}} + u_{2\bar{2}}\partial_{u_{\bar{2}2}} - u_{1\bar{1}}\partial_{u_{\bar{1}1}} - u_{1\bar{2}}\partial_{u_{\bar{2}1}} \\
& + 2(u_{22\bar{1}}\partial_{u_{\bar{1}22}} + u_{22\bar{2}}\partial_{u_{\bar{2}22}}) + u_{2\bar{1}\bar{1}}\partial_{u_{\bar{1}\bar{1}2}} + u_{2\bar{1}\bar{2}}\partial_{u_{\bar{2}\bar{1}2}} + u_{2\bar{2}\bar{2}}\partial_{u_{\bar{2}\bar{2}2}} \\
& - 2(u_{11\bar{1}}\partial_{u_{\bar{1}11}} + u_{11\bar{2}}\partial_{u_{\bar{2}11}}) - u_{1\bar{1}\bar{1}}\partial_{u_{\bar{1}\bar{1}1}} - u_{1\bar{1}\bar{2}}\partial_{u_{\bar{2}\bar{1}1}} - u_{1\bar{2}\bar{2}}\partial_{u_{\bar{2}\bar{2}1}}\big]
\end{aligned}
$$

but fortunately we shall not need the complete expressions for $\overset{3}{X}_{ij}$ and $\overset{3}{X}_{ijk}$ because further prolongation will not be necessary. We shall use these generators only to enforce the invariance conditions. Then starting with the simplest conditions and using their solutions we are allowed to simplify the expressions for the remaining complicated basis vector fields.

We shall now turn to the construction of third-order differential invariants which are annihilated by the basis vector fields. For this purpose we use the explicit expressions for the basis vector fields starting with the simplest ones

$$
\begin{aligned}
& \overset{3}{X}_{1111} = \tfrac{1}{2}\mathrm{i}\,u_2\partial_{u_{111}} \qquad \overset{3}{X}_{1122} = -\tfrac{1}{2}\mathrm{i}\,(u_1\partial_{u_{112}} - u_2\partial_{u_{122}}) \qquad \overset{3}{X}_{2222} = -\tfrac{1}{2}\mathrm{i}\,u_1\partial_{u_{222}} \\
& \overset{3}{X}_{1112} = -\tfrac{1}{2}\mathrm{i}\,(u_1\partial_{u_{111}} - u_2\partial_{u_{112}}) \qquad \overset{3}{X}_{1222} = \tfrac{1}{2}\mathrm{i}\,(u_2\partial_{u_{222}} - u_1\partial_{u_{122}})
\end{aligned} \qquad (102)
$$

and their complex conjugates. The ten operators with four indices are linear combinations of the eight partial derivative operators with respect to the third completely holomorphic, or antiholomorphic partial derivatives of u. Hence only eight of them are linearly unconnected and the rest are connected by two linear relations.

The invariance conditions require that the differential invariants Φ must be annihilated by symmetry generators X. The simplest consequence is that they must be annihilated by the four operators X_i in the unprolonged part of the symmetry generator (7) determined by equation (10). This means that the original four independent variables, namely the coordinates ζ^i cannot enter into the expressions for differential invariants explicitly. The next simplest conditions consist of

$$\overset{3}{X}_{ijkl}\Phi = 0 \qquad (103)$$

for all the vector fields (102) and their complex conjugates. It is obviously equivalent to

$$\partial_{u_{ijk}}\Phi = 0 \qquad (104)$$

that is, the invariants Φ do not depend on the eight purely holomorphic and antiholomorphic third partial derivatives u. Hence we are left with only 12 third derivatives out of their total number 20 and the total number of local coordinates which are permitted to be the arguments of invariants is equal to $39 - 12 = 27$.

Next we consider the operators $\overset{3}{X}_{ijk}$ which are simplified by equations (104),

$$\begin{aligned} \overset{3}{X}_{111} &= \tfrac{1}{2}\mathrm{i}\,(u_2\partial_{u_{11}} + u_{2\bar{1}}\partial_{u_{\bar{1}11}} + u_{2\bar{2}}\partial_{u_{\bar{2}11}}) \\ \overset{3}{X}_{112} &= -\tfrac{1}{2}\mathrm{i}\left[u_1\partial_{u_{11}} + u_{1\bar{1}}\partial_{u_{\bar{1}11}} + u_{1\bar{2}}\partial_{u_{\bar{2}11}} - (u_2\partial_{u_{12}} + u_{2\bar{1}}\partial_{u_{\bar{1}12}} + u_{2\bar{2}}\partial_{u_{\bar{2}12}})\right] \\ \overset{3}{X}_{122} &= -\tfrac{1}{2}\mathrm{i}\left[u_1\partial_{u_{12}} + u_{1\bar{1}}\partial_{u_{\bar{1}12}} + u_{1\bar{2}}\partial_{u_{\bar{2}12}} - (u_2\partial_{u_{22}} + u_{2\bar{1}}\partial_{u_{\bar{1}22}} + u_{2\bar{2}}\partial_{u_{\bar{2}22}})\right] \\ \overset{3}{X}_{222} &= -\tfrac{1}{2}\mathrm{i}\,(u_1\partial_{u_{22}} + u_{1\bar{1}}\partial_{u_{\bar{1}22}} + u_{1\bar{2}}\partial_{u_{\bar{2}22}}) \end{aligned} \tag{105}$$

and impose the conditions $\overset{3}{X}_{ijk}\Phi = 0$ together with their complex conjugates on the differential invariants. Solving these eight invariance conditions we obtain the result that the 12 third derivatives which have been left and the six second derivatives of u can enter the expressions for invariants only in the following 10 combinations:

$$\alpha = \varphi_1 \qquad \beta = \varphi_2 \qquad \gamma = \psi_1 \qquad \xi = \psi_2 \qquad \eta = \varphi u_{12\bar{2}} - \psi u_{12\bar{1}} + \mu u_{12} \tag{106}$$

together with their complex conjugates, where φ and ψ are defined by equation (24) in section 3 and subscripts on φ and ψ have the usual meaning of partial derivative with respect to coordinates. After the satisfaction of these eight conditions the number of differential invariants is reduced to 19.

We have determined that the differential invariants Φ may depend on

$$\Phi = \Phi(u, u_1, u_2, u_{\bar{1}}, u_{\bar{2}}, u_{1\bar{1}}, u_{2\bar{2}}, u_{1\bar{2}}, u_{2\bar{1}}, \alpha, \beta, \gamma, \xi, \eta, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\xi}, \bar{\eta}) \tag{107}$$

where we note that the number of independent variables in the argument of Φ above is the same as the total number of independent variables in the third prolongation after imposing $\overset{3}{X}_{ijk}\Phi = 0$ and equations (104). The remaining invariance conditions are $\overset{3}{X}_{ij}\Phi = 0$. The explicit expressions for these operators are given by

$$\begin{aligned} \widetilde{X}_{11} &= u_2\partial_{u_1} + u_{2\bar{1}}\partial_{u_{\bar{1}1}} + u_{2\bar{2}}\partial_{u_{\bar{2}1}} + \beta\partial_\alpha + \xi\partial_\gamma + \bar{\gamma}\partial_{\bar{\alpha}} + \bar{\xi}\partial_{\bar{\beta}} + \frac{1}{u_1}(u_2\eta + \psi\beta - \varphi\xi)\partial_\eta \\ \widetilde{X}_{22} &= u_1\partial_{u_2} + u_{1\bar{1}}\partial_{u_{\bar{1}2}} + u_{1\bar{2}}\partial_{u_{\bar{2}2}} + \alpha\partial_\beta + \gamma\partial_\xi + \bar{\alpha}\partial_{\bar{\gamma}} + \bar{\beta}\partial_{\bar{\xi}} + \frac{1}{u_2}(u_1\eta + \varphi\gamma - \psi\alpha)\partial_\eta \\ \widetilde{X}_{12} &= u_2\partial_{u_2} - u_1\partial_{u_1} + u_{2\bar{1}}\partial_{u_{\bar{1}2}} + u_{2\bar{2}}\partial_{u_{\bar{2}2}} - u_{1\bar{1}}\partial_{u_{\bar{1}1}} - u_{1\bar{2}}\partial_{u_{\bar{2}1}} - \alpha\partial_\alpha + \beta\partial_\beta - \gamma\partial_\gamma \\ &\quad + \xi\partial_\xi - \bar{\alpha}\partial_{\bar{\alpha}} - \bar{\beta}\partial_{\bar{\beta}} + \bar{\gamma}\partial_{\bar{\gamma}} + \bar{\xi}\partial_{\bar{\xi}} \end{aligned} \tag{108}$$

where we have dropped inessential overall factors in equations (101)

$$\overset{3}{X}_{11} = \tfrac{1}{2}\mathrm{i}\,\widetilde{X}_{11} \qquad \overset{3}{X}_{12} = \tfrac{1}{2}\mathrm{i}\,\widetilde{X}_{12} \qquad \overset{3}{X}_{22} = -\tfrac{1}{2}\mathrm{i}\,\widetilde{X}_{22} \qquad \widetilde{X}_{\bar{i}\bar{j}} = \overline{(\widetilde{X}_{ij})}$$

and transformed to the auxiliary variables (106). This transformation is well defined because, as we have already remarked, it preserves the total number of non-trivial independent variables in the third prolongation and is one-to-one. The Lie algebra of the vector fields $\widetilde{X}_{ij}$ is presented in the table 2. Here the commutators of the operators $[\widetilde{X}_{ij}, \widetilde{X}_{kl}]$ are placed at the intersection of the line and column corresponding to the operators $\widetilde{X}_{ij}$ and $\widetilde{X}_{kl}$, respectively. The result is

Table 2. Lie algebra of second-rank vector fields on $Z^3(M)$.

	$\widetilde{X}_{11}$	$\widetilde{X}_{22}$	$\widetilde{X}_{12}$
$\widetilde{X}_{11}$	0	$\widetilde{X}_{12}$	$-2\widetilde{X}_{11}$
$\widetilde{X}_{22}$	$-\widetilde{X}_{12}$	0	$2\widetilde{X}_{22}$
$\widetilde{X}_{12}$	$2\widetilde{X}_{11}$	$-2\widetilde{X}_{22}$	0

the Lie algebra of $SL(2,\mathbb{R})$. Since $\widetilde{X}_{ij}$ commute with $\widetilde{X}_{\bar{i}\bar{j}}$ the total Lie algebra is the direct product $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

Using the explicit expressions (108) we solve the six invariance conditions

$$\widetilde{X}_{ij}\Phi = 0 \tag{109}$$

together with their complex conjugates, thus reducing the number of invariants to $19-6=13$. Solving these remaining conditions for obtaining the full set of differential invariants, we obtain the explicit expressions (19)–(23) for all the functionally independent third-order differential invariants which are presented at the end of section 3.

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